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CRIX or evaluating blockchain based currencies

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The S&P500 or DAX30 are important benchmarks for the financial industry. These and other indices describe different compositions of certain segments of the financial markets. It is surprising, though, to see that emerging e-coins have not been mapped into an index yet because with cryptos like Bitcoin, a new kind of asset of great public interest has arisen. Usually, the index provider decides on a fixed number of index constituents which will represent the market segment. It is a huge challenge to set this fixed number and develop the rules to find the constituents, especially since markets change and this has to be taken into account. A method relying on the AIC is proposed to quickly react to market changes and therefore enable us to create an index, referred to as CRIX, for the cryptocurrency market. The codes used to obtain the results in this paper are available via `www.quantlet.de`.

JEL classification: C51, C52, G10

Keywords: Index construction, model selection, AIC, bitcoin, cryptocurrency

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1 Introduction

More and more companies have started offering digital payment systems. Smartphones have evolved into a digital wallet, so that it seems like we are about to enter the era of digital finance. In fact we are already inside a digital economy. The market for e-x ($x =$ "finance," "money," "book," you name it ...) has not only picked up enormous momentum but has become standard for driving innovative activities in the global economy. A few clicks at y and payment at z brings our purchase to location w. Own-currencies for the digital market were therefore just a matter of time. The idea of the Nobel Laureate Hayek, see Hayek (1990), of letting companies offer concurrent currencies seemed for a long time scarcely probable, but the invention of the Blockchain has made it possible to bring his vision to life. Cryptocurrencies (abbr. cryptos) have surfaced and opened up an angle towards this new level of economic interaction. Since the appearance of bitcoins, several new cryptos have spread through the Web and offered new ways of proliferation. Even states accept them as legal payment method or part of economic interaction. E.g., the USA classifies cryptocurrencies as commodities, Kawa (2015), and lately Japan announced that they accept them as a legal currency, EconoTimes (2016). Obviously, the crypto market is fanning out and shows clear signs of acceptance and deepening liquidity, so that a closer look at its general moves and dynamics is called for.

The technical aspects behind cryptocurrencies have been reviewed by several researchers. For a well written technical survey, see Tschorsch and Scheuermann (2015). The transaction graph of Bitcoin, the Blockchain, has received much attention too, see e.g. Ron and Shamir (2013) and Reid and Harrigan (2013). Even the economics of the Bitcoin has been studied, e.g. Kristoufek (2014). To our knowledge, the development of the entire cryptocurrency market has not been studied so far, only subsamples have been taken into account. Additionally, a reliable benchmark for this market is still missing. We will contribute to this area of research by designing a market index (benchmark) which will enable each interested party to study the performance of the crypto market or single cryptos.

First, the term benchmark has to be defined for the crypto market:

Definition 1. *A benchmark for the crypto market is a market measure which consists of a selection of representative cryptos.*

Usually index providers construct their indices, which should be constructed in terms of Definition 1, with a fixed number of index constituents, see e.g. FTSE (2016), S&P (2014) and Deutsche Boerse AG (2013). But markets change which should cause the chosen number of index constituents to be altered too. While trying to mimic the movements of an innovative market like the crypto market, one is confronted with a frequently changing market structure. This calls for a dynamic structure of the benchmark, especially for the number of constituents. The StrataQuant index family, see NYSE (2015), alters the number of constituents in each sector index dependent on their affiliation with a certain sector and membership in the Russell1000 index. But the benchmark for the crypto market won't have a parent index since it is meant to be the leading index. Therefore an independent approach is necessary. In addition to reacting to changes in the market structure, a dynamic methodology is necessary to help circumvent arbitrary rules like maximal weighting rules, MEXBOL (2013), which will preserve the diversity of an index at any time. For the crypto market what results is CRIX: a CRyptocurrency IndeX, hu.berlin/crix, which fulfills the requirement of having a dynamic structure by relying on statistical time series techniques, namely the AIC.

This paper is structured as follows. Section 2 introduces the topic and reviews the basics of index construction. In Section 3 the method for dynamic index construction is described and Section 4 introduces the remaining rules for the CRIX family. Section 5 describes the applied indices before their performance is tested in Section 6. In Sections 7 and 8 the new method is applied to the German and Mexican stock markets to test the performance of the methodology against existing indices. The codes used to obtain the results in this paper are available via `www.quantlet.de`.

2 Index construction

The basic idea of any price index is to weight the prices of its constituent goods by the quantities of the goods purchased or consumed. The Laspeyres index takes the value of a basket of k assets and compares it against a base period:

$$P^L_{0t}(k) = \frac{\sum_{i=1}^{k} P_{it} Q_{i0}}{\sum_{i=1}^{k} P_{i0} Q_{i0}} \tag{1}$$

with P_{it} the price of asset i at time t and Q_{i0} the quantity of asset i at time 0 (the base period). For market indices, such as CRSP, S&P500 or DAX, the quantity Q_{i0} is the number of shares of the asset i in the base period. Multiplied with its corresponding price, there results the market capitalization of a company, which implies that the constituents of the index are weighted by their market capitalizations. But markets change. A company which was representative for market developments in the 1990s might no longer be important today. On top of that, companies can go bankrupt, a corporation can raise the number of its outstanding shares, or trading in it can become infrequent. All these situations must produce a change in the index structure, so that the market is still adequately represented. This means that the company has to drop out of the index and has to be replaced by a suitable one. The index rules determine in which cases such an event happens. But the formula of Laspeyres (1) can not handle such events because a change of constituents will result in a change in the index value that is not due to price changes. Therefore, established price indices like DAX or S&P500, see Deutsche Boerse AG (2013) and S&P (2014) respectively, and the newly founded index CRIX(k), a CRyptocurrency IndeX, hu.berlin/crix, use the adjusted formula of Laspeyres,

$$\text{CRIX}_t(k) = \frac{\sum_{i=1}^{k} P_{it} Q_{it}}{Divisor(k)_t} \tag{2}$$

with P, Q and i defined as before. The *Divisor* ensures that the index value of CRIX has a predefined value on the starting date. It is defined as

$$Divisor(k)_0 = \frac{\sum_{i=1}^{k} P_{i0} Q_{i0}}{\text{starting value}}. \tag{3}$$

The starting value could be any possible number, commonly 100, 1000 or 10000. It ensures that a positive or negative development from the base period will be revealed. Whenever changes to the structure of CRIX occur, the *Divisor* is adjusted in such a way that only price changes are reflected by the index. Defining k_1 and k_2 as number of constituents, it results

$$\frac{\sum_{i=1}^{k_1} P_{i,t-1}Q_{i,t-1}}{Divisor(k_1)_{t-1}} = \text{CRIX}_{t-1}(k_1) = \text{CRIX}_t(k_2) = \frac{\sum_{j=1}^{k_2} P_{j,t}Q_{j,t}}{Divisor(k_2)_t}. \tag{4}$$

In indices like FTSE, S&P500 or DAX the number of index members is fixed, $k_1 = k_2$, see FTSE (2016), S&P (2014) and Deutsche Boerse AG (2013). As long as the goal behind these indices is the reflection of the price development of the selected assets, this is a straightforward approach. But, e.g., DAX is also meant to be an indicator for the development of the market as a whole, see Janßen and Rudolph (1992). This raises automatically the question of whether the included assets are representative of the market. Since the constituents are chosen using a top-down approach, meaning that the biggest companies by market capitalization are included, the intuitive answer is yes. But maybe more assets are necessary to describe the market appropriately. One may answer that total market indices like the Wilshire 5000, S&P Total Market Index or CRSP U.S. Total Market Index, see Wilshire Associates (2015), S&P (2015) and CRSP (2015), are able to provide a full description. But media reporting has shown that the smaller indices like DAX and S&P500 receive more attention in evaluating the movements of their corresponding markets. Each interested party may have different reasons for preferring smaller indices to total market indices. We feel that it is appealing to know which are the representative assets in a market. Additionally, we are concerned that a huge index would include illiquid and non-investable assets. This raises the question which value of k is optimal for building a benchmark for the market. Additionally, especially young and innovative markets may change their structure over time. Therefore, the goal of this paper will be to develop a methodology which is able to find an accurate and dynamic benchmark for a market which is as sparse with constituents as possible. Since the cryptocurrency (crypto) market shows a frequently changing market structure with a huge number of illiquid cryptos, we apply the methodology directly to this market.

3 Dynamic index construction

This section is dedicated to describing the composition rule which is used to find the number of index members—the spine of CRIX. Since CRIX will be a benchmark for the crypto market, the dimension and evaluation of the market has to be defined, called the total market:

Definition 2. *The total market (TM) consists of all cryptos in the crypto universe. Its value is the combined market value of the cryptos.*

To compare the TM with a benchmark candidate, it will be normalized by a Divisor,

$$\text{TM}(K)_t = \frac{\sum_{i=1}^{K} P_{it}Q_{it}}{Divisor(K)} \tag{5}$$

with K the number of all cryptos in the crypto universe.

The goal is to optimize k so that a sparse and accurate solution to

$$\min_k \varepsilon(k)_t = \text{TM}(K)_t - \text{CRIX}(k)_t, \tag{6}$$
$$\text{s.t.: } 1 \leq k \leq k^u$$
$$k^u \in [1, K]$$

will be found where $\text{CRIX}(k)_t$ is the CRIX with k constituents at time point t and ε_t is the difference from the total market. To solve the problem (6), a loss function has to be defined first. Since the goal is finding an accurate benchmark, a squared loss function is a good choice because it penalizes far apart solutions for (6) stronger than close ones. The expected squared loss is defined as

$$\mathsf{E}(\|\varepsilon(k)_t\|^2|k) \tag{7}$$

where $\|\cdot\|^2$ is the squared Euclidean norm.

The density, f, which is contained in the expectation operator E, is estimated non-parametrically with an Epanechnikov kernel, since according to Härdle et al. (2004) the Epanechnikov (1969) kernel shows a good balance between variance optimization and numerical performance. The bandwidth selection is performed with the plug-in selector, mentioned in Sheather and Jones (1991) and further described in Wand and Jones (1994).

Sparsity is a further part of the goal because we are concerned with including small and illiquid cryptos in the benchmark which will not have an effect on the market index and which are difficult to invest in. Due to the construction of the optimization procedure, the fit of (6) will become better with higher k when a certain number of constituents has already been reached. With small values of k, the fit may not enhance, due to lacking a parameter estimation. It follows that the penalization technique for determining the sufficient k has to be powerful enough or otherwise the representative benchmark will contain many cryptos. Even small, illiquid and barely representative cryptos for the market may enter the crypto index. The analysis in Chapter 6 shows exactly this result for the index Exact Full CRIX (EFCRIX), see Chapter 5 for its description. Therefore, we decided to include this constraint into the goal of the index construction to ensure that just a small number of representative assets form the benchmark at any time.

Since the value of $\text{TM}(K)_t$ is unknown and not measurable due to a lack of information, the total market index will be defined and used as a proxy for the TM. The definition is inspired by total market indices like CRSP (2015), S&P (2015) and Wilshire Associates (2015). They use all stocks for which prices are available.

Definition 3. *The total market index (TMI) contains all cryptos in the crypto universe for which prices are available. The cryptos are weighted by their market capitalization.*

This changes (5) to

$$\text{TMI}_t(k_{max}) = \frac{\sum_{i=1}^{k_{max}} P_{it}Q_{it}}{Divisor(k_{max})}$$

with k_{max} the maximum number of cryptos with available prices and (6) to

$$\begin{aligned} \min_k \widehat{\varepsilon}(k)_t = \text{TMI}(k_{max})_t - \text{CRIX}(k)_t \\ \text{s.t.: } 1 \le k \le k^u \\ k^u \in [1, k_{max}]. \end{aligned} \tag{8}$$

To solve a time series optimization problem for which a squared loss function is used and a penalization for the number of model parameters is requested, a huge number of criteria would be applicable. We discuss the ones mentioned in Droge (2006). This paper covers cross validation (CV), full cross validation (FCV), Generalized Cross Validation (GCV), Generalized Full Cross Validation (GFCV), Mallows' C_p, Akaike's Final Prediction Error (FPE), Shibata (SH), AIC, BIC and Hannan Quinn (HQ). The first one, CV, see Stone

(1974), is a widely used criterion in practice for the mean squared error of prediction aims, see Droge (2006). It is defined as

$$\mathrm{CV}(k) = T^{-1} \sum_{i=1}^{T} \{\mathrm{TMI}_t(k_{max}) - \mathrm{CRIX}(k)_{-t}\}^2 \tag{9}$$

where $\mathrm{CRIX}(k)_{-t}$ is the estimate of $\mathrm{CRIX}(k)$ without the observation t. CV does not work in this context for two reasons. First, the derivation of (9) requires an estimation but in the derivation of CRIX no parameter estimation is necessary, compare (2). Second, CV does not involve any penalty for the number of constituents. This criterion is not applicable, since the goal of sparsity in k requests a penalty for the number of constituents. The FCV criterion, see Bunke et al. (1999), is not applicable for the same reasons. The GCV criterion, see Craven and Wahba (1978), is defined as

$$\mathrm{GCV}(k) = \frac{T^{-1} \sum_{t=1}^{T} \widehat{\varepsilon}(k)_t^2}{(1 - T^{-1}k)^2} \tag{10}$$

by assuming that $k < T$, see Droge (2006). According to Arlot and Celisse (2010), the asymptotic optimality of GCV was shown in several frameworks. The GFCV was introduced by Droge (1996) and is defined to be

$$\mathrm{GFCV}(k) = T^{-1} \sum_{t=1}^{T} \widehat{\varepsilon}(k)_t^2 (1 + T^{-1}k)^2. \tag{11}$$

A further score, SH,

$$\mathrm{SH}(k) = \frac{T + 2k}{T^2} \sum_{t=1}^{T} \widehat{\varepsilon}(k)_t^2, \tag{12}$$

was shown to be asymptotically optimal, Shibata (1981), and asymptotically equivalent to Mallows' C_p, FPE and AIC.

Mallows (1973)' C_p:

$$\mathrm{C}_p(k) = \frac{\sum_{t=1}^{T} \widehat{\varepsilon}(k)_t^2}{\sigma(k)^2} - T + 2 \cdot k \tag{13}$$

with $\sigma(k)^2$ the variance of $\{\widehat{\varepsilon}(k)_t\}$ with $t = 1, \ldots, T$. $C_p(k)$ tends to choose models which overfit and is not consistent in selecting the true model, see Mallick and Yi (2013), Woodroofe (1982) and Nishii (1984).

The FPE uses the formula

$$\mathrm{FPE}(k) = \frac{T + k}{(T - k)T} \sum_{t=1}^{T} \widehat{\varepsilon}(k)_t^2, \tag{14}$$

see Akaike (1970). So far, the discussed criteria depend on little data information. Just the squared residuals and, in the case of Mallows' C_p, the variance are taken into account. The further criteria use more information by depending on the maximum likelihood, derived by

$$L(k) = \max_{\theta} \prod_{t} f(\widehat{\varepsilon}_t, \theta, k), \tag{15}$$

where f, in (7), represents the density of the $\widehat{\varepsilon}(k)_t$ over all t and θ are model parameters.

The first one is the AIC which is defined to be

$$\text{AIC}(k) = -2 \log L(k) + k \cdot 2, \tag{16}$$

Akaike (1998). If the true model is of finite dimension, then neither FPE nor AIC are consistent, compare Hurvich and Tsai (1989). But Shibata (1983) showed the asymptotic efficiency of Mallows' C_p, FPE and AIC under the assumption of an infinite number of regression variables or an increasing number of regression variables with the sample size.

On the other hand, the BIC, defined as

$$\text{BIC}(k) = -2 \log L(k) + k \cdot \log(T), \tag{17}$$

see Schwarz (1978), is consistent in choosing the true model, Nishii (1984). A further consistent criterion is the one proposed by Hannan and Quinn (1979), defined as

$$\text{HQ} = -2 \log L(k) + 2k \cdot \log\{\log(T)\}. \tag{18}$$

We'll evaluate now which criteria to use for our purpose. Since CRIX is to be a benchmark model, all possible models under certain restrictions for the number of parameters are included in the test set, $\Theta_{AIC} = \{\text{CRIX}(k_1), \text{CRIX}(k_2), \ldots, \text{TMI}\}$, where $k_1, k_2, \ldots$ are predefined values. Recall that the intention behind CRIX is to discover the best model to describe the data (benchmark) under a squared loss function.

Define the loss function in (7) for $\widehat{\varepsilon}(k)$,

$$R_T(k) = \mathsf{E}(||\text{TMI}(k_{max})_t - \text{CRIX}(k)_t||^2|\text{TMI}), \tag{19}$$

and define the number of constituents which minimize the risk in $R_T(k)$ as k^* for the model set Θ, Shibata (1983). k^* will be interpreted as the number of constituents which balance the bias and variance, define

$$H_T(k) = ||\text{TMI}(k_{max})_t - \mathsf{E}(\text{CRIX}(k)_t)||^2 + k\sigma(k)^2. \tag{20}$$

Mean efficiency shall be defined as

$$\text{eff}(\Theta) = H_T(k^*)/R_T(\Theta), \tag{21}$$

see Shibata (1983). A criteria is defined to be asymptotic mean efficient if

$$\text{a.eff}(\Theta) = \liminf_{T \to \infty} H_T(k^*)/R_T(\Theta) = 1 \tag{22}$$

This result holds if the number of constituents k increases with T, Shibata (1983). This assumption is plausible in this case since longer time horizons T would include cryptos which aren't part of shorter ones due to bankruptcy or since they haven't been found yet. Both leads to more complexity. It follows that all of the asymptotically optimal criteria would lead to a mean efficient model choice in terms of squared risk for a given selection of models which fits the intention to discover a best model. It remains to find the suitable one.

Define the characteristic function as

$$\varphi(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx \tag{23}$$

with $i \in C$ and $t \in R$. The Fourier inversion theorem states (Shephard (1991)):

Theorem 1. *Suppose g and φ are integrable in the Lebesgue sense and*

$$\varphi(t) = \int_{-\infty}^{\infty} \exp(itx) g(x) dx, \tag{24}$$

then

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \varphi(t) dt. \tag{25}$$

holds everywhere.

The moment generating function is defined as

$$M(t) = \int_{-\infty}^{\infty} \exp(tx) f(x) dx. \tag{26}$$

If the moments generating function exists, it holds

$$\varphi(t) = M(it). \tag{27}$$

We see that the characteristic function depends on the moment generating function of $\widehat{\varepsilon}$. Therefore, knowing all moments of a distribution means knowing the distribution itself. Most of the asymptotically optimal criteria depend on the empirical versions of the first two moments of $\widehat{\varepsilon}$. Just the AIC uses the full distribution and therefore all the moments. This makes its information basis richer. For the derivation of the number of index members of CRIX, we will use the AIC, because it uses the most information compared to the other asymptotically optimal criteria: it is the only one which depends on the likelihood.

To decide with AIC which number k should be used, a procedure was created which compares the difference between the TMI, see Definition 3, and several candidate indices,

$$\widehat{\varepsilon}(k_j)_t = \text{TMI}(k_{max})_t - \text{CRIX}(k_j)_t, \tag{28}$$

where $\text{CRIX}(k_j)_t$ is the CRIX version with k_j constituents and $\widehat{\varepsilon}(k_j)_t$ is the respective difference. The candidate indices, $\text{CRIX}(k_j)$, have different numbers of constituents which fulfill $k_1 < k_2 < k_3 < \cdots$. By definition both information criteria evaluate the differences, $\widehat{\varepsilon}(k_j)_t$, between the candidates and the TMI with the respective maximum likelihood $L(k_j)$, see Equations (16) and (17). Usually, this involves an estimation of the parameters, θ, in terms of maximizing the likelihood function. But in this case, the information about the parameters is already known. For each constituent, the weight is equal to 1. Due to this prior information and the formula for the definition of CRIX, see (2), all the parameters are restricted to be equal 1. This implies that we are dealing automatically with a maximum likelihood for each model.

Since the differences between the $\text{TMI}(k_{max})$ and $\text{CRIX}(k_j)_t$ are caused over time by the missing time series in $\text{CRIX}(k_j)_t$, the independence assumption can not be fulfilled by construction. But Györfi et al. (1989) give arguments that under certain conditions, the rate of convergence is essentially the same as for an independent sample. Since the same data are used to estimate f_j and the information criterion, a "leave-one-out" cross-validation procedure is performed in order to have independence between the calculation of the density and the information criterion; see Potapov et al. (2011). The 5-step procedure works as follows:

1. Create T datasets $\widehat{\varepsilon}_{-t} = \{\ldots, \widehat{\varepsilon}_{t-1}, \widehat{\varepsilon}_{t+1}, \ldots\}$, leaving out $\widehat{\varepsilon}_t$.

2. Compute the Kernel Density Estimator (KDE) for each $\widehat{\varepsilon}_{-t}$.
3. Compute the log likelihood (16) for $\widehat{\varepsilon}_t$ with KDE for $\widehat{\varepsilon}_{-t}$.
4. Sum the log likelihoods.
5. Calculate the AIC.

The next section describes the further index rules for CRIX.

4 CRIX family rules

To ensure that an index provides a good proxy of the market, its constituents need to change their values in a representative way. This implies that their liquidity has to be high enough to represent changes in the market. The dynamic rule will already filter out small cryptos which have less impact on the market index, but we will introduce a further rule to strengthen the purpose of excluding illiquid cryptos since the dynamic rule may have difficulties with illiquid cryptos with a high market capitalization. Known market indices like DAX and S&P500 rely on a measure which takes into account for the free floating trading volume. The use of this approach for the cryptocurrency market is questionable because the number of coins which are held as a long term investment is unknown for a crypto due to the anonymous owner structure. A simple approach would be to assume that all coins are free floating since they belong to a currency and there should not be any interest in holding them as is the case for stocks, where the shares regulate the ownership structure of a company. But data for Bitcoin show that most of the coins are not used. A different approach is necessary. This will be, for CRIX, a combination of the liquidity rules from the STOXX Japan 600, see STOXX (2015), and the AEX family, compare Euronext (2014). One defines a crypto to be eligible for the CRIX if it fulfills one of the two following rules:

1. 0.25 percentile of ADTV (Average Daily Trading Volume in USD):

$$\text{ADTV}_i \geq \text{ADTV}_{0.25}$$

2. 0.25 percentile of ADRTC (Average Daily Relative Traded Coins):

$$\text{ADRTC}_i \geq \text{ADRTC}_{0.25}$$

The rules were chosen so that CRIX adapt dynamically to the market. A fixed minimum value for the ADTV and ADRTC would have the disadvantage that it would have to be adjusted from time to time to ensure that it would still well represent the market situation. Furthermore it is worth looking at both measures since certain cryptos may have a very low trading volume in USD because they are just not as worthy at the moment, yet when the trading on the exchanges is high, they might be an important crypto for the crypto community, so that they should not be excluded from the circle of eligible cryptos just because the ADTV is small.

The constituents of the indices are regularly looked over so that the corresponding index always represents its asset universe well. It is common to do this on a quarterly basis, see e.g. Deutsche Boerse AG (2013), MEXBOL (2013) and S&P (2014). In case of CRIX this reallocation is much faster. In the past, coins have shown a very volatile behavior, not just in the manner of price volatility. In some weeks, many occur out of nothing in the market and many others vanish from the market even when they were before very important, e.g., Auroracoin. This calls for a faster reallocation of the market benchmark than on a quarterly basis. We choose a monthly reallocation to make sure that CRIX catches the momentum of the cryptocurrency market well. Therefore, on the last day of every month, which cryptos had the highest market capitalization on the last day in the last month will be checked and the first k such will be included in CRIX for the coming month. Formally speaking, with $J = \{P_{i,t_m} Q_{i,t_m}\}$ for $i = 1, \dots, K$ with $|J| = K$, one has the following for the chosen set j_k:

$$j_b = \{j_{b-1} \cup i : P_{i,t_m} Q_{i,t_m} \geq a, \forall a \in J \backslash j_{b-1}\} \quad \text{for} \quad b = 1, \dots, k \qquad (29)$$
$$j_0 = \emptyset$$

where t_m is the last day of the last month. Established indices like DAX often use the cumulative market capitalization of the last month or months for the evaluation. We decided to use a different approach since the information criteria selection procedure depends on the market capitalization on the last day in the last month, see Section 10, especially the derivation (40). One should note that this causes the selection of the index members to depend on possibly unusual divergences, since just one data point is taken into account.

Since a review of an index is commonly performed on a quarterly basis, see, e.g., Deutsche Boerse AG (2013) or S&P (2014), the number of index members of CRIX will be checked on a quarterly basis too. The described procedure from Section 3 will be applied to the observations from the last three months on the last day of the third month after the markets closed. The number of index constituents, k, will be used for the next three months.

It may happen that some data are missing for some of the analyzed time series. If an isolated missing value occurs alone in the dataset, meaning that the values before and after it are not missing, then Missing At Random (MAR) is assumed. This assumption means that just observed information cause the missingness, Horton and Kleinman (2007). The Last-Observation-Carried-Forward (LOCF) method is then applied to fill the gap for the application of the AIC. We did not choose a different approach since a regression or imputation may alter the data in the wrong direction. By LOCF, we imply no change and just do not exclude the crypto. If two or more data are missing in a row, then the MAR assumption may be violated, therefore no method is applied. The corresponding time series is then excluded from the computation in the derivation period. If data are missing during the computation of the index values, the LOCF method is applied too. This is done to make the index insensitive to this crypto at this time point. CRIX should mimic market changes, therefore an imputation or regression method for the missing data would distort the view of the market.

5 The CRIX family

Using the described methods and rules from above, three indices will be proposed. This indices provide a different look at the market.

1. CRIX:
   The first and leading index is CRIX. While the choice for the best number of

constituents is made, their numbers are chosen in steps of five. It is common in practice to construct market indices with a number of constituents which is evenly divisible by five, see e.g. FTSE (2016), S&P (2014), Deutsche Boerse AG (2013). Therefore this choosing is performed for CRIX(k), $k = 5, 10, 15, \ldots$. Since the global minimum for the AIC criterion may involve many index constituents, but a sparse index is the goal, the search for the optimal model terminates at level j whenever

$$\text{AIC}(k_{j-1}) < \text{AIC}(k_j). \tag{30}$$

Therefore merely a local optimum will be achieved in most of the cases for $\Theta = \Theta_{AIC}$, in (21). But the choice is still asymptotically optimal by defining $\Theta = \{\Theta_{AIC} | k_i \leq k_j \forall i\}$. In Section 6 it will be shown that the performance of the index is already very good.

2. ECRIX:
The second constructed index is called Exact CRIX (ECRIX). It follows the above rules too except for the liquidity rule. For ECRIX, no liquidity restriction is applied. Also, the number of its constituents is chosen in steps of 1. Therefore the set of models contains CRIX(k), $k = 1, 2, 3, \ldots$.

3. EFCRIX:
Since the decision procedures for CRIX and ECRIX terminate when the AIC rises for the first time, Exact Full CRIX will be constructed to visualize whether the decision procedure works fine for the two covered indices. The intention is to have an index which may approach the TMI but only in case even small assets help improve the view of the total market, a benchmark for the benchmarks. It'll be derived with the AIC procedure, compare Section 3. The decision rule is based on

$$\min_{k_j} \text{AIC}(k_j) \tag{31}$$

for $\Theta = \Theta_{AIC}$, in (21). This index computes the AIC for every possible number of constituents and what is chosen is the number where the AIC becomes minimal. Again, no liquidity rule is applied.

6 Performance analysis

The indices CRIX, ECRIX, EFCRIX have been proposed to give insight into the crypto market. Our RDC crypto database covers data for 290 cryptocurrencies, kindly provided by CoinGecko. The data used for the analysis cover daily closing data for prices, market volume and market capitalization in USD for each crypto in the time period from 2014-04-01 to 2016-04-06. Crypto exchanges are open on the weekends, therefore data for weekend closing prices exist. Since crypto exchanges do not finish trading after a certain time point every day, a time point which serves as a closing time has to be defined. CoinGecko used 12 am UTC time zone. One should note that missing data are observed in the dataset, therefore the last rules from Chapter 4 will come into play.

Figure 1 shows the performance of CRIX, and Figure 2 the differences between CRIX and both ECRIX and EFCRIX. For the purpose of comparison, the indices were recalibrated on the recalculation dates since the index constituents change then. We do not provide each index plot individually since they perform almost equally. However, the AIC method

Figure 1: Performance of CRIX

CRIXindex CRIXcode

gave very different numbers of constituents for the corresponding indices. The numbers of constituents are given in Table 2. Apparently the methodology of EFCRIX causes its number of constituents to become close to maximal in every period. ECRIX has mostly much fewer constituents than CRIX and EFCRIX due to the fact that this index just runs until a local optimum.

	MSE	MDA
CRIX vs. TMI	2.0687	0.9935
ECRIX vs. TMI	9.2370	0.9870
EFCRIX vs. TMI	0.0444	1.0000

Table 1: Comparison of CRIX, ECRIX, EFCRIX against TMI

The performance of the AIC method in selecting the number of constituents for CRIX is shown in the boxplots in Figures 6 and 7 in the Appendix, see 10.2. The method steps further, if the Inner Quartile Range is shrinking and the number of outliers is diminishing. Both are an indicator for a shrinking variance. If we encounter the opposite finding, the method stops. Obviously, the method greatly depends on its ability to lower the variance of the $\widehat{\varepsilon}(k)_t$, which was shown in the Appendix for an example of the normal distribution, see Chapter 10.1.

Since the indices CRIX and ECRIX are just optimized until a local optimum, they are expected to perform less optimally than the EFCRIX against the TMI. Table 1 gives the

Period	CRIX	ECRIX	EFCRIX	Maximum achievable
1	5	3	40	41
2	25	8	119	119
3	5	12	170	170
4	30	10	190	190
5	15	2	204	205
6	30	8	215	215
7	55	4	214	214

Table 2: Number of constituents in respective periods

CRIXmembers

Mean Square Error (MSE) and the Mean Directional Accuracy (MDA), defined as

$$\text{MSE}\{\text{CRIX}(k)\} = \frac{1}{T}\sum_{t=1}^{T}\{\text{CRIX}(k)_t - \text{TMI}(k_{\max})_t\}^2 \tag{32}$$

$$\begin{aligned}\text{MDA}\{\text{CRIX}(k)\} = \frac{1}{T}\sum_{t=1}^{T} 1[&\text{sign}\{\text{TMI}(k_{\max})_t - \text{TMI}(k_{\max})_{t-1}\} \\ &= \text{sign}\{\text{CRIX}(k)_t - \text{CRIX}(k)_{t-1}\}]\end{aligned} \tag{33}$$

where 1 is the indicator function and sign$(\cdot)$ gives the sign of the respective equation. The recalibration of the indices on the recalculation date is important for the computation of the MSE, since altering the constituents may change the future development in terms of MSE. The MDA is insensitive to the recalibration. Apparently EFCRIX performs best, which can be explained due to its larger number of index constituents. The CRIX, ECRIX and EFCRIX are close in terms of the MDA but the MSE is better for EFCRIX. Additionally, ECRIX performs worse than CRIX in terms of MSE and MDA. At the same time, the number of constituents is higher for CRIX than for ECRIX except in one period, see Table 2.

CRIX was constructed with steps of five which is common in practice, but this analysis showed that ECRIX would work well for the crypto market too. Additionally, the analysis showed that it is indeed unnecessary to choose the global optimal AIC. Even a local optimum and a much lower number of constituents is able to mimic the market movements very well in terms of the MDA. Furthermore, even for ECRIX there was more than one constituent selected at any time. This shows that Bitcoin, which currently clearly dominates the market in terms of market capitalization and trading volume, doesn't lead the market. Other cryptocurrencies are important for the market movements too.

7 Application to the German stock market

The CRIX methodology was derived with the idea of finding a method which allows to mimicking young and fast changing markets appropriately. But well known major markets usually change their structure too. We tested the proposed methodology on the German stock market, which has four major indices: DAX, MDAX, SDAX and TecDAX. The DAX could be used to determine the overall market direction, see Janßen and Rudolph (1992), which classifies DAX as a benchmark for the German stock market. Since it is chosen from the so called prime segment, it has some prior restrictions. We would like to see with

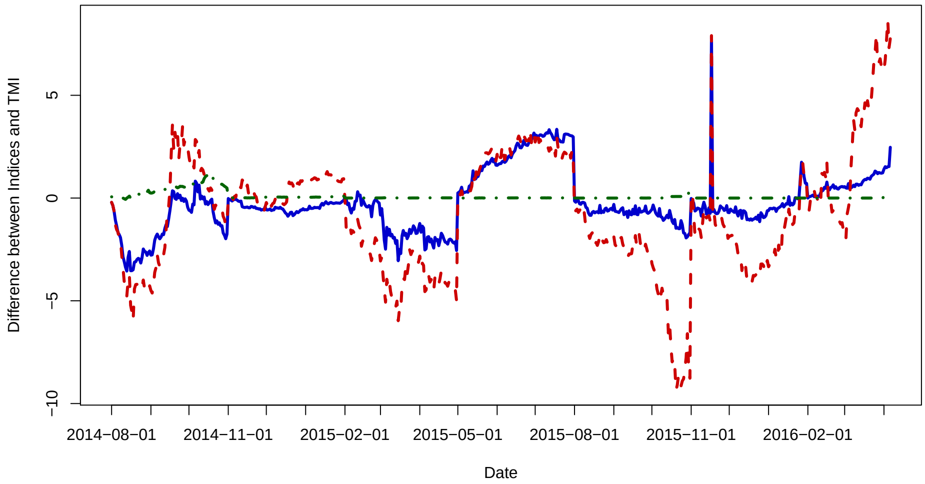

Figure 2: Realized difference between TMI and CRIX (solid), ECRIX (dashed), EFCRIX (dotdashed)

CRIXfamdiff CRIXcode

our methodology whether DAX is an adequate benchmark for the total market. Following Definition 3, we define all available stocks as the TMI and apply our new method to find an appropriate index. Again, the 5-step method from Section 3 was applied to find the number of constituents, but it starts at 30 members to check if more constituents are necessary. The method for the identification of k is applied yearly and the reallocation of the included assets is performed quarterly, like DAX. k is chosen on a yearly basis, because the general index maintenance date for DAX is on a yearly basis, too. To be in line with the DAX reallocation dates, the index calculation will start after the third Friday of September and the reallocation dates are the third Fridays of December, March, June and September, see Deutsche Boerse AG (2013).

The data were fetched from Datastream in the period 20000616 until 20151218. We took all stocks which are German companies and are traded on XETRA. Any time series for which Datastream reported an error either for the price or market capitalization data was excluded from the analysis. The index, computed with the new methodology, is called Flexible DAX (FDAX). One should note that the analysis starts three months after the starting point of the dataset due to the initialization period of FDAX.

The figure 3 shows the number of members of FDAX and DAX in the respective periods. Most of the time, the number of index constituents for FDAX is higher than the 30 members of DAX. Just around 2004-2005 is the k more frequently 30. One might hint that a higher reported variability in one period should cause an increase in k in the next period, since it was shown that the selection method depends on the variance, see Section 10. The figure (4) shows that this idea can partially be supported. The derivation of the conditional variance was performed with a GARCH(1,1) model and the daily results were summed as in Liu and Tse (2013). The GARCH model was found and further described in Bollerslev (1986). Obviously, in the extreme cases increases the k in the next period, see 2001 and 2011.

The computation of the MSE and MDA, see Table 3, shows that FDAX is a more

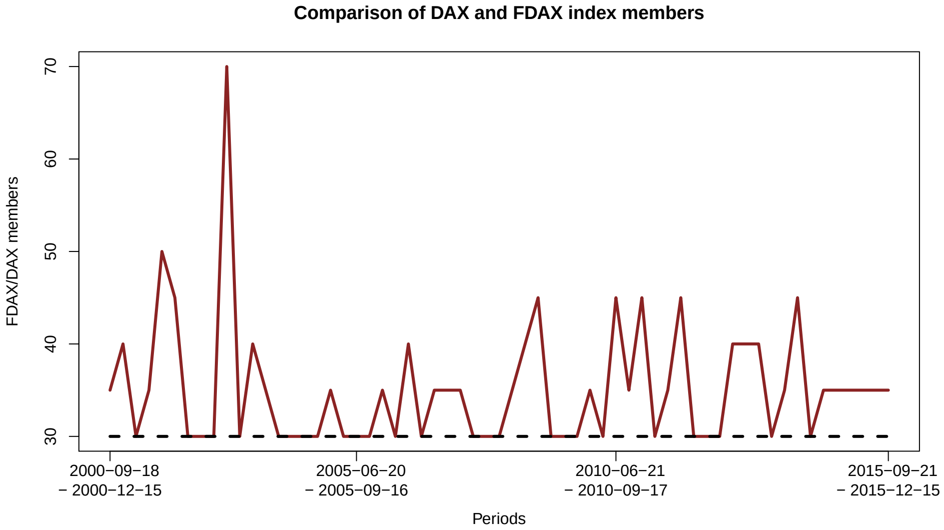

Figure 3: Number of constituents of DAXCRIX and DAX in the respective periods
CRIXdaxmembers CRIXcode

accurate benchmark for the total market as DAX. Since Janßen and Rudolph (1992) state that DAX may be used to analyze the movements of the total market, an MDA of 91 % is indeed good. But FDAX mimics the market even better, with a MDA of 95 %. Also the MSE for FDAX is less than half of the one of DAX. Therefore the methodology fulfilled its goal to find a sparse and accurate benchmark, depending on the MDA. But if a further goal is to find an index which will be eligible for investment, like many major stock indices, we would not recommend applying the proposed methodology directly. The swapping of the index members will cause the transaction costs to rise, which may prevent the investor from gaining profits. If not as an investment underlying, FDAX could be used to find the maximal k from the observed data and use it for a certain time as the number of index constituents. In the case of DAX in this 15 year sample, would this be 70 index constituents, which could be used for the next 15 years. As the analysis just showed, also a k of 30 would give a desired result, which classifies DAX as an accurate and investable benchmark for the German stock market. But FDAX is more favorable.

	MSE	MDA
FDAX vs. TMI	347.20	0.95
DAX vs. TMI	756.47	0.91

Table 3: Comparison of DAX with CRIX methodology (FDAX) and rescaled DAX against TMI

8 Application to Mexican stock market

The Mexican stock market is represented by the IPC35, MEXBOL (2013). One of its rules is a readjustment of the weights to lower the effect of dominant stocks. In the crypto

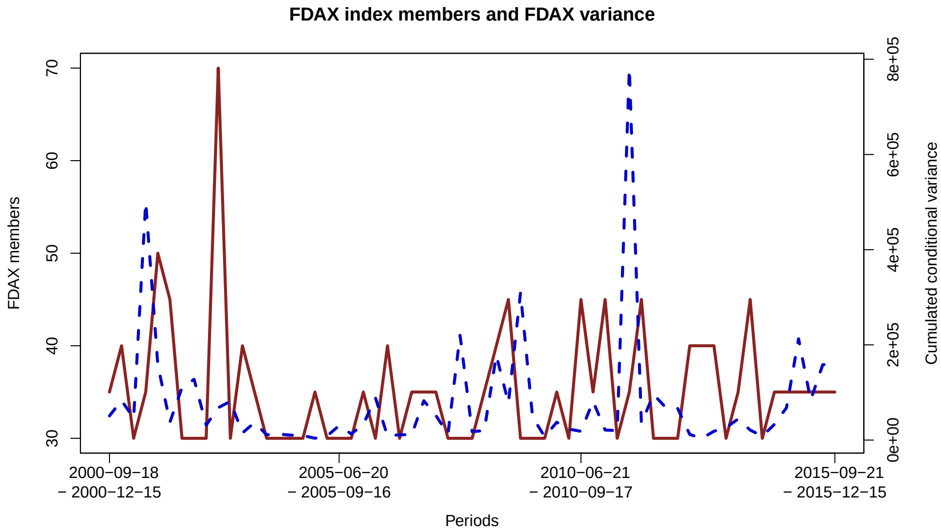

Figure 4: Number of constituents of FDAX (solid) and cumulated variance of FDAX (dashed)

CRIXdaxmembersvar

market Bitcoin is such a dominant asset. The CRIX methodology could help to circumvent arbitrary rules and develop an index to represent the market accurately.

The data were fetched from Datastream for the period 19960601 until 20150529 and cover all Mexican companies listed in Datastream. The specifications of the methodology are the same as for the German stock market except for the recalculation date. In line with the methodology of the IPC35, we recalculated the index with the closing data of the last business days of August, November, February and May, therefore the recalculated index starts on the first business days of September, December, March and June. The TMI will be all fetched companies. The choice of k starts with 35 since this is the amount of constituents of IPC.

Again, the CRIX methodology works well. The MSE is very low compared to the one for the IPC35 and the MDA gives a much better performance too, see Table 4. We can conclude that the methodology helped to circumvent the usage of arbitrary rules for the weights in the rules of the indices and enhances at the same time the performance of the market index. Figure (5) shows the number of index members of the FIPC compared to the IPC. Obviously, the methodology also suggests using more than 35 index members most of the time which is the number of members of the IPC.

	MSE	MDA
FIPC vs. TMI	242.07	0.97
IPC vs. TMI	151113.43	0.91

Table 4: Comparison of IPC with CRIX methodology (FIPC) and rescaled IPC against TMI

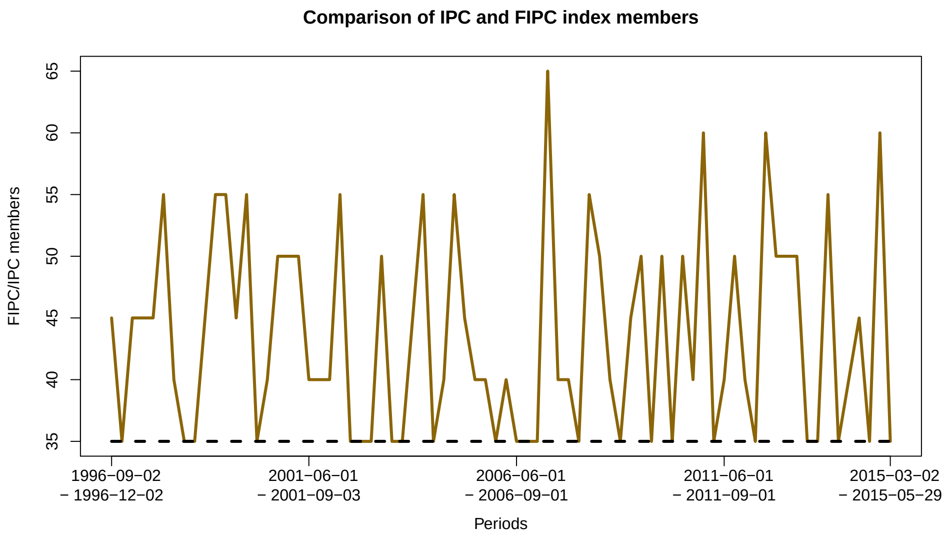

Figure 5: Number of constituents of FIPC (solid) and IPC (dashed) in the respective periods

CRIXipcmembers CRIXcode

9 Conclusion

An index construction for cryptocurrencies requires a new methodology to find the right number of index members. Innovative markets, like the one for cryptocurrency's, change their structure frequently. The proposed methods were applied to oracle a new family of indices, which are displayed and updated on a daily basis on `hu.berlin/crix`. The performance of the new indices were studied and it was shown that the dynamic AIC based methodology results in indices with stable properties. The results show that a market like the crypto market - momentarily dominated by Bitcoin - still needs a representative index since Bitcoin does not lead the market. The AIC based method was also applied to the German stock market. The results yield a more accurate benchmark in terms of MDA. In applying the CRIX methodology to the Mexican stock market one finds high accuracy of it in terms of MSE and MDA.

The CRIX technology enhances the construction of an index if the goal is to find a sparse and accurate benchmark. But since the methodology is based on a changing parameter k, a portfolio based on the index may have high transaction costs. A possibility would be to identify, in a short term analysis, the number k which gives an enhanced fit in the long run. This would expand the usage of the methodology to investment oriented indices too, since a fixed optimal k would lower transaction costs and ensure a good fit in calm and crisis situations at the same time.

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10 Appendix

10.1 Influences on the AIC methodology

The information criterion based methodology depends on the variance of the index members. For clarification, assume normally distributed error terms: $\varepsilon(k) \sim \mathrm{N}(0, \sigma(k)^2)$. Then

$$\log L(k) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\sigma(k)^2 - \frac{1}{2\sigma(k)^2}\sum_{i=1}^{T}\varepsilon(k)_i^2. \tag{34}$$

In the case of estimation, denote $RSS(k) = \sum_{i=1}^{T}\widehat{\varepsilon}(k)_i^2$ and $\hat{\sigma}(k)^2 = T^{-1}RSS(k)$. Then

$$\begin{aligned}
\log L(k) &= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log T^{-1}RSS(k) - \frac{1}{2T^{-1}RSS(k)}RSS(k) &(35)\\
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log T^{-1}RSS(k) - \frac{T}{2} &(36)\\
&= -\frac{T}{2}\log T^{-1}RSS(k) + C &(37)
\end{aligned}$$

with $C = -\frac{T}{2}\log(2\pi) - \frac{T}{2}$. Since C does not depend on any model parameters, just on the data length T, this part of the equation could be omitted.

$$\begin{aligned}
\mathrm{AIC}(k) &= T\log T^{-1}RSS(k) + 2\cdot k &(38)\\
&= T\log\widehat{\sigma}(k)^2 + 2\cdot k &(39)
\end{aligned}$$

The enhancement in the fit to the Total Market Index (TMI) by adding more constituents, k, determines the degree of improvement of the likelihood.

Of interest is the difference between the TMI and CRIX. Define $a = \frac{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}}{\sum_{i=1}^{k} P_{i0}Q_{i0}}$, for which it holds that $1 \leq a \leq \frac{k_{max}}{k}$. Only the first period for starting the index will be covered. The results hold for each decision periods. Since CRIX started with a value of

1000, this starting value will be used.

$$\begin{aligned}
\widehat{\varepsilon}(k) &= \text{TMI}(k_{max})_t - \text{CRIX}(k)_t \\
&= \text{CRIX}(k_{max})_t - \text{CRIX}(k)_t \\
&= \sum_{i=1}^{k_{max}} P_{it}Q_{it}/Divisor(k_{max}) - \sum_{j=1}^{k} P_{jt}Q_{jt}/Divisor(k) \\
&= \frac{\sum_{i=1}^{k_{max}} P_{it}Q_{it}}{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000} - \frac{\sum_{i=1}^{k} P_{it}Q_{it}}{\sum_{i=1}^{k} P_{i0}Q_{i0}/1000} \\
&= \frac{\sum_{i=1}^{k_{max}} P_{it}Q_{it} - \frac{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}}{\sum_{i=1}^{k} P_{i0}Q_{i0}} \sum_{i=1}^{k} P_{it}Q_{it}}{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000} \\
&= \frac{\sum_{i=1}^{k_{max}} P_{it}Q_{it} - a\sum_{i=1}^{k} P_{it}Q_{it}}{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000} \\
&= \frac{\sum_{i=k+1}^{k_{max}} P_{it}Q_{it} + (1-a)\sum_{i=1}^{k} P_{it}Q_{it}}{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000}
\end{aligned}$$

Assume joint normality between the P_i and derive $\widehat{\sigma}(k)^2$, where Cov is the covariance operator. It holds that

$$\sigma_t(k)^2 = \text{Var}\{\widehat{\varepsilon}(k)\} \tag{40}$$

$$\begin{aligned}
&= \text{Var}\{\frac{\sum_{i=k+1}^{k_{max}} P_{it}Q_{it} + (1-a)\sum_{i=1}^{k} P_{it}Q_{it}}{\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000}\} \\
&= \frac{\text{Var}\{\sum_{i=k+1}^{k_{max}} P_{it}Q_{it}\} + (1-a)^2\text{Var}\{\sum_{i=1}^{k} P_{it}Q_{it}\}}{(\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000)^2} \\
&+ \frac{(1-a)\text{Cov}\{\sum_{i=k+1}^{k_{max}} P_{it}Q_{it}, \sum_{i=1}^{k} P_{it}Q_{it}\}}{(\sum_{i=1}^{k_{max}} P_{i0}Q_{i0}/1000)^2}
\end{aligned} \tag{41}$$

10.2 Boxplots of CRIX error terms

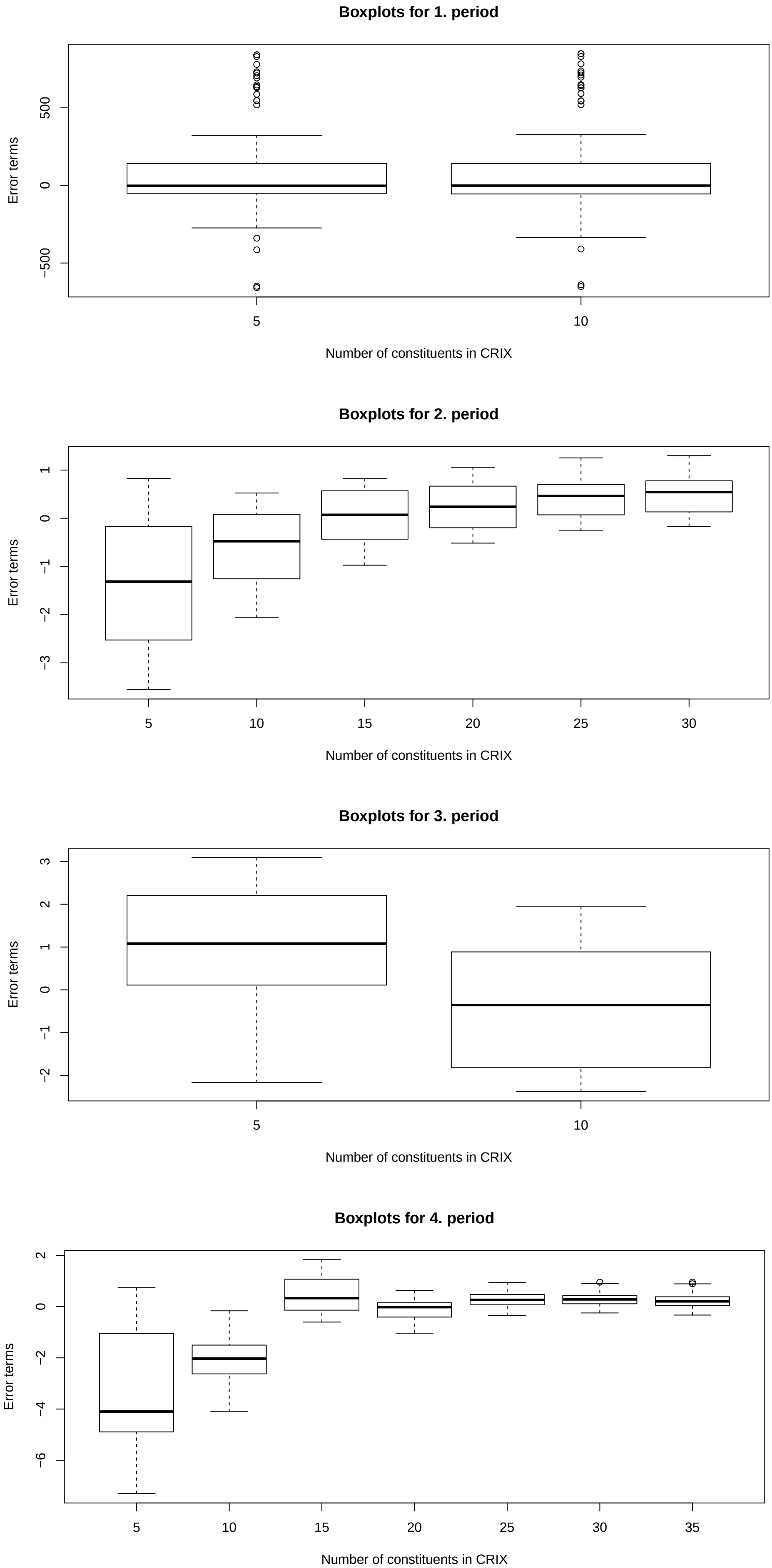

Figure 6: Boxplots for the first 4 periods of CRIX error terms

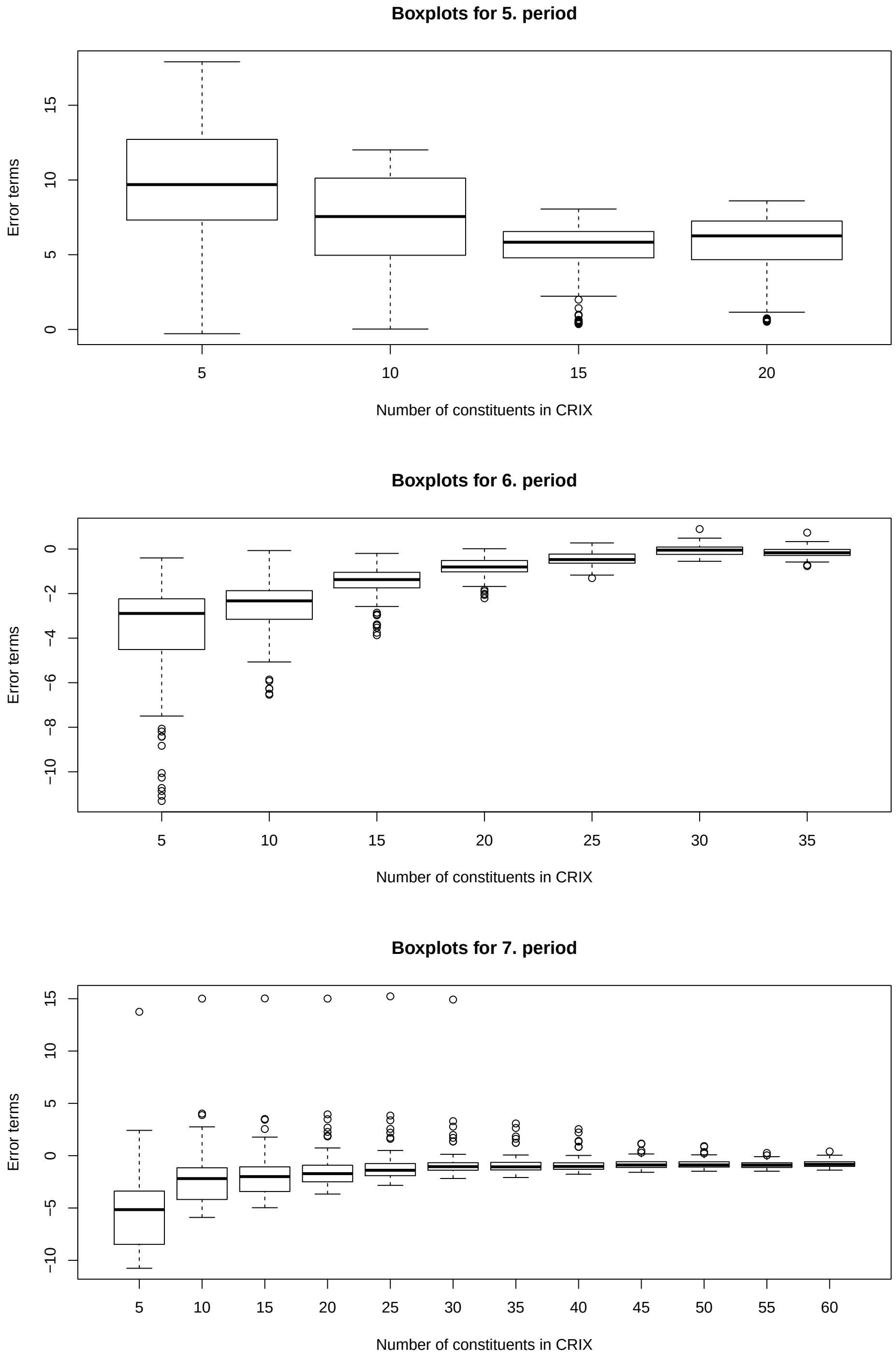

Figure 7: Boxplots for the last 3 periods of CRIX error terms

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